

# Exercise Solutions for *Introduction to 3D Game Programming with DirectX 11*

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Frank Luna

## Solutions to Part I

### Chapter 1

1. Let  $\mathbf{u} = (1, 2)$  and  $\mathbf{v} = (3, -4)$ . Perform the following computations and draw the vectors relative to a 2D coordinate system.

- a)  $\mathbf{u} + \mathbf{v}$
- b)  $\mathbf{u} - \mathbf{v}$
- c)  $2\mathbf{u} + \frac{1}{2}\mathbf{v}$
- d)  $-2\mathbf{u} + \mathbf{v}$

**Solution:**

- a)  $(1, 2) + (3, -4) = (1 + 3, 2 + (-4)) = (4, -2)$
- b)  $(1, 2) - (3, -4) = (1, 2) + (-3, 4) = (1 - 3, 2 + 4) = (-2, 6)$
- c)  $2(1, 2) + \frac{1}{2}(3, -4) = (2, 4) + \left(\frac{3}{2}, -2\right) = \left(\frac{7}{2}, 2\right)$
- d)  $-2(1, 2) + (3, -4) = (-2, -4) + (3, -4) = (1, -8)$

2. Let  $\mathbf{u} = (-1, 3, 2)$  and  $\mathbf{v} = (3, -4, 1)$ . Perform the following computations.

- a)  $\mathbf{u} + \mathbf{v}$
- b)  $\mathbf{u} - \mathbf{v}$
- c)  $3\mathbf{u} + 2\mathbf{v}$
- d)  $-2\mathbf{u} + \mathbf{v}$

**Solution:**

- a)  $(-1, 3, 2) + (3, -4, 1) = (-1 + 3, 3 + (-4), 2 + 1) = (2, -1, 3)$
- b)  $(-1, 3, 2) - (3, -4, 1) = (-1, 3, 2) + (-3, 4, -1) = (-4, 7, 1)$
- c)  $3(-1, 3, 2) + 2(3, -4, 1) = (-3, 9, 6) + (6, -8, 2) = (3, 1, 8)$
- d)  $-2(-1, 3, 2) + (3, -4, 1) = (2, -6, -4) + (3, -4, 1) = (5, -10, -3)$

3. This exercise shows that vector algebra shares many of the nice properties of real numbers (this is not an exhaustive list). Assume  $\mathbf{u} = (u_x, u_y, u_z)$ ,  $\mathbf{v} = (v_x, v_y, v_z)$ , and  $\mathbf{w} = (w_x, w_y, w_z)$ . Also assume that  $c$  and  $k$  are scalars. Prove the following vector properties.

- a)  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$  (Commutative Property of Addition)
- b)  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$  (Associative Property of Addition)
- c)  $(ck)\mathbf{u} = c(k\mathbf{u})$  (Associative Property of Scalar Multiplication)
- d)  $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$  (Distributive Property 1)
- e)  $\mathbf{u}(k + c) = k\mathbf{u} + c\mathbf{u}$  (Distributive Property 2)

**Solution:**

a)

$$\begin{aligned}\mathbf{u} + \mathbf{v} &= (u_x, u_y, u_z) + (v_x, v_y, v_z) \\ &= (u_x + v_x, u_y + v_y, u_z + v_z) \\ &= (v_x + u_x, v_y + u_y, v_z + u_z) \\ &= (v_x, v_y, v_z) + (u_x, u_y, u_z) \\ &= \mathbf{v} + \mathbf{u}\end{aligned}$$

b)

$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_x, u_y, u_z) + ((v_x, v_y, v_z) + (w_x, w_y, w_z)) \\ &= (u_x, u_y, u_z) + (v_x + w_x, v_y + w_y, v_z + w_z) \\ &= (u_x + (v_x + w_x), u_y + (v_y + w_y), u_z + (v_z + w_z)) \\ &= ((u_x + v_x) + w_x, (u_y + v_y) + w_y, (u_z + v_z) + w_z) \\ &= (u_x + v_x, u_y + v_y, u_z + v_z) + (w_x, w_y, w_z) \\ &= ((u_x, u_y, u_z) + (v_x, v_y, v_z)) + (w_x, w_y, w_z) \\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}\end{aligned}$$

c)

$$\begin{aligned}(ck)\mathbf{u} &= (ck)(u_x, u_y, u_z) \\ &= ((ck)u_x, (ck)u_y, (ck)u_z) \\ &= (c(ku_x), c(ku_y), c(ku_z)) \\ &= c(ku_x, ku_y, ku_z) \\ &= c(k\mathbf{u})\end{aligned}$$

d)

$$\begin{aligned}k(\mathbf{u} + \mathbf{v}) &= k((u_x, u_y, u_z) + (v_x, v_y, v_z)) \\ &= k(u_x + v_x, u_y + v_y, u_z + v_z)\end{aligned}$$

$$\begin{aligned}
&= (k(u_x + v_x), k(u_y + v_y), k(u_z + v_z)) \\
&= (ku_x + kv_x, ku_y + kv_y, ku_z + kv_z) \\
&= (ku_x, ku_y, ku_z) + (kv_x, kv_y, kv_z) \\
&= k\mathbf{u} + k\mathbf{v}
\end{aligned}$$

e)

$$\begin{aligned}
\mathbf{u}(k + c) &= (u_x, u_y, u_z)(k + c) \\
&= (u_x(k + c), u_y(k + c), u_z(k + c)) \\
&= (ku_x + cu_x, ku_y + cu_y, ku_z + cu_z) \\
&= (ku_x, ku_y, ku_z) + (cu_x, cu_y, cu_z) \\
&= k\mathbf{u} + c\mathbf{u}
\end{aligned}$$

4. Solve the equation  $2((1, 2, 3) - \mathbf{x}) - (-2, 0, 4) = -2(1, 2, 3)$  for  $\mathbf{x}$ .

**Solution:** Use vector algebra to solve for  $\mathbf{x}$ :

$$\begin{aligned}
2((1, 2, 3) - \mathbf{x}) - (-2, 0, 4) &= -2(1, 2, 3) \\
(2, 4, 6) - 2\mathbf{x} + (2, 0, -4) &= (-2, -4, -6) \\
-2\mathbf{x} + (2, 0, -4) &= (-4, -8, -12) \\
-2\mathbf{x} &= (-6, -8, -8) \\
\mathbf{x} &= (3, 4, 4)
\end{aligned}$$

5. Let  $\mathbf{u} = (-1, 3, 2)$  and  $\mathbf{v} = (3, -4, 1)$ . Normalize  $\mathbf{u}$  and  $\mathbf{v}$ .

**Solution:**

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 3^2 + 2^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \left( -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right)$$

$$\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2 + 1^2} = \sqrt{9 + 16 + 1} = \sqrt{26}$$

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left( \frac{3}{\sqrt{26}}, -\frac{4}{\sqrt{26}}, \frac{1}{\sqrt{26}} \right)$$

6. Let  $k$  be a scalar and let  $\mathbf{u} = (u_x, u_y, u_z)$ . Prove that  $\|k\mathbf{u}\| = |k|\|\mathbf{u}\|$ .

$$\|k\mathbf{u}\| = \sqrt{(ku_x)^2 + (ku_y)^2 + (ku_z)^2} = \sqrt{k^2(u_x^2 + u_y^2 + u_z^2)} = |k| \sqrt{u_x^2 + u_y^2 + u_z^2} = |k|\|\mathbf{u}\|$$

7. Is the angle between  $\mathbf{u}$  and  $\mathbf{v}$  orthogonal, acute, or obtuse?

- a)  $\mathbf{u} = (1, 1, 1), \mathbf{v} = (2, 3, 4)$   
 b)  $\mathbf{u} = (1, 1, 0), \mathbf{v} = (-2, 2, 0)$   
 c)  $\mathbf{u} = (-1, -1, -1), \mathbf{v} = (3, 1, 0)$

- a)  $\mathbf{u} \cdot \mathbf{v} = 1(2) + 1(3) + 1(4) = 9 > 0 \Rightarrow$  acute  
 b)  $\mathbf{u} \cdot \mathbf{v} = 1(-2) + 1(2) + 0(0) = 0 \Rightarrow$  orthogonal  
 c)  $\mathbf{u} \cdot \mathbf{v} = -1(3) + (-1)(1) + (-1)(0) = -4 < 0 \Rightarrow$  obtuse

8. Let  $\mathbf{u} = (-1, 3, 2)$  and  $\mathbf{v} = (3, -4, 1)$ . Find the angle  $\theta$  between  $\mathbf{u}$  and  $\mathbf{v}$ .

**Solution:** Using the equation  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos \theta$  we have:

$$\begin{aligned} \theta &= \cos^{-1} \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \right) \\ &= \cos^{-1} \left( \frac{-1(3) + 3(-4) + 2(1)}{\sqrt{14}\sqrt{26}} \right) \\ &= \cos^{-1} \left( \frac{-13}{\sqrt{14}\sqrt{26}} \right) \\ &= 132.95^\circ \end{aligned}$$

9. Let  $\mathbf{u} = (u_x, u_y, u_z), \mathbf{v} = (v_x, v_y, v_z)$ , and  $\mathbf{w} = (w_x, w_y, w_z)$ . Also let  $c$  and  $k$  be scalars. Prove the following dot product properties.

- a)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$   
 b)  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$   
 c)  $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$   
 d)  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$   
 e)  $\mathbf{0} \cdot \mathbf{v} = 0$

**Solution:**

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (u_x, u_y, u_z) \cdot (v_x, v_y, v_z) \\ &= u_x v_x + u_y v_y + u_z v_z \\ &= (v_x, v_y, v_z) \cdot (u_x, u_y, u_z) \\ &= \mathbf{v} \cdot \mathbf{u} \end{aligned}$$

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) &= (u_x, u_y, u_z) \cdot (v_x + w_x, v_y + w_y, v_z + w_z) \\ &= u_x(v_x + w_x) + u_y(v_y + w_y) + u_z(v_z + w_z) \\ &= u_x v_x + u_x w_x + u_y v_y + u_y w_y + u_z v_z + u_z w_z \\ &= (u_x v_x + u_y v_y + u_z v_z) + (u_x w_x + u_y w_y + u_z w_z) \\ &= \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \end{aligned}$$

$$\begin{aligned} k(\mathbf{u} \cdot \mathbf{v}) &= k(u_x v_x + u_y v_y + u_z v_z) \\ &= (ku_x)v_x + (ku_y)v_y + (ku_z)v_z \end{aligned}$$

$$\begin{aligned}
&= (k\mathbf{u}) \cdot \mathbf{v} \\
&= u_x(kv_x) + u_y(kv_y) + u_z(kv_z) \\
&= \mathbf{u} \cdot (k\mathbf{v})
\end{aligned}$$

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{v} &= v_x v_x + v_y v_y + v_z v_z \\
&= \sqrt{v_x^2 + v_y^2 + v_z^2}^2 \\
&= \|\mathbf{v}\|^2
\end{aligned}$$

$$\mathbf{0} \cdot \mathbf{v} = 0v_x + 0v_y + 0v_z = 0$$

10. Use the law of cosines ( $c^2 = a^2 + b^2 - 2ab \cos \theta$ , where  $a$ ,  $b$ , and  $c$  are the lengths of the sides of a triangle and  $\theta$  is the angle between sides  $a$  and  $b$ ) to show

$$u_x v_x + u_y v_y + u_z v_z = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

*Hint:* Consider Figure 1.9 and set  $c^2 = \|\mathbf{u} - \mathbf{v}\|^2$ ,  $a^2 = \|\mathbf{u}\|^2$  and  $b^2 = \|\mathbf{v}\|^2$ , and use the dot product properties from the previous exercise.

**Solution:**

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cos \theta \\
\|\mathbf{u} - \mathbf{v}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\
(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\
\mathbf{u} \cdot \mathbf{u} - 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v} &= \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2\|\mathbf{u}\| \|\mathbf{v}\| \cos \theta \\
\mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta
\end{aligned}$$

11. Let  $\mathbf{n} = (-2, 1)$ . Decompose the vector  $\mathbf{g} = (0, -9.8)$  into the sum of two orthogonal vectors, one parallel to  $\mathbf{n}$  and the other orthogonal to  $\mathbf{n}$ . Also, draw the vectors relative to a 2D coordinate system.

**Solution:**

$$\mathbf{g}_{\parallel} = \text{proj}_{\mathbf{n}}(\mathbf{g}) = \frac{(\mathbf{g} \cdot \mathbf{n})}{\|\mathbf{n}\|^2} \mathbf{n} = \frac{-9.8}{5} (-2, 1) = -1.96(-2, 1) = (3.92, -1.96)$$

$$\mathbf{g}_{\perp} = \mathbf{g} - \mathbf{g}_{\parallel} = (0, -9.8) - (3.92, -1.96) = (-3.92, -7.84)$$

12. Let  $\mathbf{u} = (-2, 1, 4)$  and  $\mathbf{v} = (3, -4, 1)$ . Find  $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ , and show  $\mathbf{w} \cdot \mathbf{u} = 0$  and  $\mathbf{w} \cdot \mathbf{v} = 0$ .

**Solution:**

$$\begin{aligned}
\mathbf{w} = \mathbf{u} \times \mathbf{v} &= (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x) \\
&= (1 + 16, 12 + 2, 8 - 3)
\end{aligned}$$

$$= (17, 14, 5)$$

$$\mathbf{w} \cdot \mathbf{u} = 17(-2) + 14(1) + 5(4) = -34 + 14 + 20 = 0$$

$$\mathbf{w} \cdot \mathbf{v} = 17(3) + 14(-4) + 5(1) = 51 - 56 + 5 = 0$$

13. Let the following points define a triangle relative to some coordinate system:  $\mathbf{A} = (0, 0, 0)$ ,  $\mathbf{B} = (0, 1, 3)$ , and  $\mathbf{C} = (5, 1, 0)$ . Find a vector orthogonal to this triangle. *Hint:* Find two vectors on two of the triangle's edges and use the cross product.

**Solution:**

$$\mathbf{u} = \mathbf{B} - \mathbf{A} = (0, 1, 3)$$

$$\mathbf{v} = \mathbf{C} - \mathbf{A} = (5, 1, 0)$$

$$\begin{aligned} \mathbf{n} &= \mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x) \\ &= (0 - 3, 15 - 0, 0 - 5) \\ &= (-3, 15, -5) \end{aligned}$$

14. Prove that  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$ . *Hint:* Start with  $\|\mathbf{u}\|\|\mathbf{v}\| \sin \theta$  and use the trigonometric identity  $\cos^2 \theta + \sin^2 \theta = 1 \Rightarrow \sin \theta = \sqrt{1 - \cos^2 \theta}$ ; then apply Equation 1.4.

**Solution:**

To make the derivation simpler, we compute the following three formulas up front:

$$\begin{aligned} \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 &= (u_x^2 + u_y^2 + u_z^2)(v_x^2 + v_y^2 + v_z^2) \\ &= u_x^2(v_x^2 + v_y^2 + v_z^2) + u_y^2(v_x^2 + v_y^2 + v_z^2) + u_z^2(v_x^2 + v_y^2 + v_z^2) \\ &= u_x^2 v_x^2 + u_x^2 v_y^2 + u_x^2 v_z^2 + u_y^2 v_x^2 + u_y^2 v_y^2 + u_y^2 v_z^2 + u_z^2 v_x^2 + u_z^2 v_y^2 + u_z^2 v_z^2 \end{aligned} \quad (1)$$

$$\begin{aligned} (\mathbf{u} \cdot \mathbf{v})^2 &= (u_x v_x + u_y v_y + u_z v_z)(u_x v_x + u_y v_y + u_z v_z) \\ &= u_x v_x(u_x v_x + u_y v_y + u_z v_z) + u_y v_y(u_x v_x + u_y v_y + u_z v_z) \\ &\quad + u_z v_z(u_x v_x + u_y v_y + u_z v_z) \\ &= u_x v_x u_x v_x + u_x v_x u_y v_y + u_x v_x u_z v_z + u_y v_y u_x v_x + u_y v_y u_y v_y \\ &\quad + u_y v_y u_z v_z + u_z v_z u_x v_x + u_z v_z u_y v_y + u_z v_z u_z v_z \\ &= u_x^2 v_x^2 + 2u_x v_x u_y v_y + 2u_x v_x u_z v_z + u_y^2 v_y^2 + 2u_y v_y u_z v_z + u_z^2 v_z^2 \end{aligned} \quad (2)$$

$$\begin{aligned} \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 &= u_x^2 v_y^2 + u_x^2 v_z^2 + u_y^2 v_x^2 + u_y^2 v_z^2 + u_z^2 v_x^2 + u_z^2 v_y^2 \\ &\quad - 2u_x v_x u_y v_y - 2u_x v_x u_z v_z - 2u_y v_y u_z v_z \\ &= (u_y^2 v_z^2 - 2u_y v_y u_z v_z + u_z^2 v_y^2) + (u_z^2 v_x^2 - 2u_x v_x u_z v_z + u_x^2 v_z^2) \\ &\quad + (u_x^2 v_y^2 + -2u_x v_x u_y v_y + u_y^2 v_x^2) \end{aligned} \quad (3)$$

Now,

$$\begin{aligned}
 \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1 - \cos^2\theta} \\
 &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\
 &= \|\mathbf{u}\|\|\mathbf{v}\|\sqrt{\frac{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\
 &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}
 \end{aligned}$$

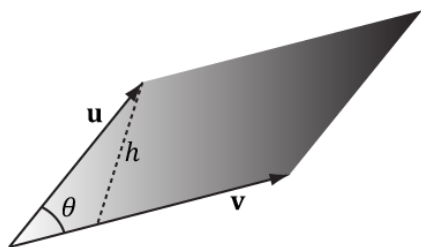
And

$$\begin{aligned}
 \|\mathbf{u} \times \mathbf{v}\| &= \|\langle u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x \rangle\| \\
 &= \sqrt{(u_y v_z - u_z v_y)^2 + (u_z v_x - u_x v_z)^2 + (u_x v_y - u_y v_x)^2} \\
 &= \sqrt{(u_y^2 v_z^2 - 2u_y v_z u_z v_y + u_z^2 v_y^2) + (u_z^2 v_x^2 - 2u_z v_x u_x v_z + u_x^2 v_z^2) + (u_x^2 v_y^2 - 2u_x v_y u_y v_x + u_y^2 v_x^2)} \\
 &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}
 \end{aligned}$$

Thus we obtain the desired result:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta$$

15. Prove that  $\|\mathbf{u} \times \mathbf{v}\|$  gives the area of the parallelogram spanned by  $\mathbf{u}$  and  $\mathbf{v}$ ; see Figure below.



**Solution:**

The area is the base times the height:

$$A = \|\mathbf{v}\|h$$

Using trigonometry, the height is given by  $h = \|\mathbf{u}\|\sin(\theta)$ . This, along with the application of Exercise 14, we can conclude:

$$A = \|\mathbf{u}\|\|\mathbf{v}\|\sin(\theta) = \|\mathbf{u} \times \mathbf{v}\|$$

16. Give an example of 3D vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  such that  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ . This shows the cross product is generally not associative. *Hint:* Consider combinations of the simple vectors  $\mathbf{i} = (1, 0, 0)$ ,  $\mathbf{j} = (0, 1, 0)$ , and  $\mathbf{k} = (0, 0, 1)$ .

**Solution:**

Choose  $\mathbf{u} = (1, 1, 0)$ ,  $\mathbf{v} = \mathbf{i}$ , and  $\mathbf{w} = \mathbf{j}$ . Then:

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (1, 1, 0) \times \mathbf{k} = (1, -1, 0)$$

But,

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$$

17. Prove that the cross product of two nonzero parallel vectors results in the null vector; that is,  $\mathbf{u} \times k\mathbf{u} = \mathbf{0}$ . *Hint:* Just use the cross product definition.

**Solution:**

$$\begin{aligned} \mathbf{u} \times k\mathbf{u} &= (u_y k u_z - u_z k u_y, u_z k u_x - u_x k u_z, u_x k u_y - u_y k u_x) \\ &= (k u_y u_z - k u_z u_y, k u_z u_x - k u_x u_z, k u_x u_y - k u_y u_x) \\ &= \mathbf{0} \end{aligned}$$

18. Orthonormalize the set of vectors  $\{(1, 0, 0), (1, 5, 0), (2, 1, -4)\}$  using the Gram-Schmidt process.

**Solution:**

Let  $\mathbf{v}_0 = (1, 0, 0)$ ,  $\mathbf{v}_1 = (1, 5, 0)$ , and  $\mathbf{v}_2 = (2, 1, -4)$ .

Set

$$\mathbf{w}_0 = (1, 0, 0)$$

Then

$$\text{proj}_{\mathbf{w}_0}(\mathbf{v}_1) = (\mathbf{w}_0 \cdot \mathbf{v}_1)\mathbf{w}_0 = (1, 0, 0)$$

$$\begin{aligned} \mathbf{x}_1 &= (1, 5, 0) - \text{proj}_{\mathbf{w}_0}(\mathbf{v}_1) \\ &= (1, 5, 0) - (1, 0, 0) \\ &= (0, 5, 0) \end{aligned}$$



$$\mathbf{w}_1 = \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = (0,1,0)$$

$$\text{proj}_{\mathbf{w}_0}(\mathbf{v}_2) = (\mathbf{w}_0 \cdot \mathbf{v}_2)\mathbf{w}_0 = (2,0,0)$$

$$\text{proj}_{\mathbf{w}_1}(\mathbf{v}_2) = (\mathbf{w}_1 \cdot \mathbf{v}_2)\mathbf{w}_1 = (0,1,0)$$

$$\begin{aligned}\mathbf{x}_2 &= (2, 1, -4) - \text{proj}_{\mathbf{w}_0}(\mathbf{v}_2) - \text{proj}_{\mathbf{w}_1}(\mathbf{v}_2) \\ &= (2, 1, -4) - (2,0,0) - (0,1,0) \\ &= (0, 0, -4)\end{aligned}$$

$$\mathbf{w}_2 = \frac{\mathbf{x}_2}{\|\mathbf{x}_2\|} = (0, 0, -1)$$

It is clear that the set  $\{(1, 0, 0), (0,1,0), (0, 0, -1)\}$  is orthonormal.